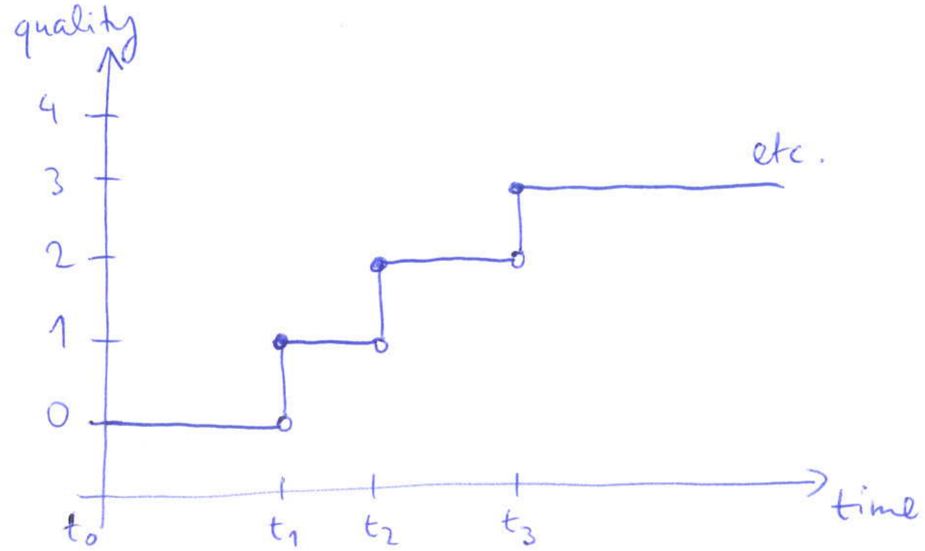
The Schumpeterian (quality ladder) growth model

- KEY INSIGHT: long-run growth driven by quality improvements within a predefined set of product varieties

- SCHUMPETERIAN "CREATIVE DESTRUCTION"



- timing of quality innovations is uncertain
- productivity in a given sector is given by q^{κ}, where $q > 1$, and κ — rung in the quality ladder
- the researcher responsible for a quality improvement retains the monopoly right to produce the good

$\boxed{!}$ → only the highest available grade of goods will actually be produced in equilibrium

- THERE ARE INFINITELY MANY SECTORS, INDEXED BY $i \in [0,1]$
  - this helps overcome aggregate uncertainty (LAW OF LARGE NUMBERS)

The quality adjusted amount of i-th intermediate input:

$$\tilde{X}_i = q^{k_i} X_i .$$

The aggregate production function is

$$Y = A X^{\alpha} L_Y^{1-\alpha}, \quad \text{where } X = \left(\int_0^1 (q^{k_i} x_i)^{\theta} di \right)^{\frac{1}{\theta}}, \quad \theta \in (0,1).$$

We maintain the Dixit-Stiglitz monopolistic competition setup.

1°/ Final goods producers' problem

$$\max_{\{x_i\}, L_Y} \left\{ Y - \int_0^1 q_i x_i \, di - w L_Y \right\}, \quad \text{using prod. fcts defined above}$$

$$\frac{\partial \Pi}{\partial L_Y} = (1-\alpha) \frac{Y}{L_Y} - w = 0 \implies \underbrace{w = (1-\alpha) \frac{Y}{L_Y}}$$

$$\frac{\partial \Pi}{\partial x_i} = \frac{\alpha}{\theta} \left(\int_0^1 (q^{k_i} x_i)^{\theta} di \right)^{\frac{\alpha}{\theta}-1} A L_Y^{1-\alpha} \theta q^{\theta k_i} x_i^{\theta - 1} - q_i = 0$$

$$\implies q_i = \alpha \frac{Y}{X^{\theta}} q^{\theta k_i} x_i^{\theta-1} \implies \underbrace{x_i(q_i) = \left(\frac{\alpha Y q^{\theta k_i}}{q_i X^{\theta}} \right)^{\frac{1}{1-\theta}}}_{\text{DEMAND CURVE FOR INTERMEDIATE GOODS}}$$

2°/ Intermediate goods producers' problem, $i \in [0,1]$

- it is assumed the marginal cost of production is 1

$$\pi(k_i) = (q_i - 1)x_i = (q_i - 1)\, x_i(q_i).$$

- monopoly pricing

$$\frac{\partial \pi(k_i)}{\partial q_i} = x_i(q_i) + (q_i - 1)\, x_i'(q_i) = \underbrace{x_i(q_i)}_{>0}\left(1 - \frac{1}{1-\theta}\,\frac{q_i - 1}{q_i}\right) = 0$$

hence we obtain

$$\frac{1}{1-\theta}\,\frac{q_i - 1}{q_i} = 1 \Rightarrow \boxed{q_i = \frac{1}{\theta},} \quad \text{for all } i \in [0,1].$$

3°/ By symmetry,

$$\bar{q} = q_i = q_j, \quad \text{for all } i, j.$$

However, output may vary because sectors can be at different rungs of the quality ladder:

- $$x_i = \left(\frac{\alpha Y}{X^\theta}\,\theta\, q^{\theta k_i}\right)^{\frac{1}{1-\theta}} = (\alpha\theta Y)^{\frac{1}{1-\theta}} \cdot \left(\frac{q^{k_i}}{X}\right)^{\frac{\theta}{1-\theta}}.$$

Profits are given by

$$\pi(k_i) = \left(\frac{1-\theta}{\theta}\right)\bar{\pi} \cdot \left(q^{k_i}\right)^{\frac{\theta}{1-\theta}}, \quad \text{where } \bar{\pi} = \left(\frac{\alpha\theta Y}{X^\theta}\right)^{\frac{1}{1-\theta}}.$$

4°/ Value of a quality innovation

- monopoly rights are perpetual
- value of these rights falls to zero when a new quality rung is attained within the sector.

- The present value of profits for the inventor of rung κ_i:

$$V(\kappa_i) = \int_{t_{\kappa_i}}^{t_{\kappa_i+1}} \pi(\kappa_i)\, e^{-\bar{r}(v, t_{\kappa_i})(v - t_{\kappa_i})}\, dv,$$

where $\bar{r}(v, t_{\kappa_i}) := \frac{1}{v - t_{\kappa_i}} \int_{t_{\kappa_i}}^{v} r(\omega)\, d\omega$ is the average interest rate between t_{κ_i} and v.

- If the interest rate is fixed, this simplifies to:

$$V(\kappa_i) = \pi(\kappa_i) \cdot \frac{1 - e^{r(t_{\kappa_i+1} - t_{\kappa_i})}}{r}.$$

5°/ Aggregation (so far)

Def. Quality index of the economy is

$$Q \equiv \left(\int_0^1 q^{\kappa_i \left(\frac{\theta}{1-\theta}\right)} di \right)^{\frac{1-\theta}{\theta}}$$

Using the definition of X, we have

$$X^\theta = \int_0^1 (q^{\kappa_i} x_i)^\theta di = \int_0^1 \left(q^{\kappa_i \left(1 + \frac{\theta}{1-\theta}\right)} \left(\frac{\alpha \theta Y}{X^\theta} \right)^{\frac{1}{1-\theta}} \right)^\theta di =$$

$$= \left(\frac{\alpha \theta Y}{X^\theta} \right)^{\frac{\theta}{1-\theta}} \underbrace{\int_0^1 q^{\kappa_i \left(\frac{\theta}{1-\theta}\right)} di}_{Q^{\frac{\theta}{1-\theta}}}.$$

Hence, $X = (\alpha \theta Y)^{\frac{1}{1-\theta}} X^{-\frac{\theta}{1-\theta}} Q^{\frac{1}{1-\theta}} \Rightarrow \underline{X = \alpha \theta Y Q}$

Inserting into the production function,

$$Y = A (\alpha \theta Y Q)^\alpha L_Y^{1-\alpha} \Rightarrow Y^{1-\alpha} = A (\alpha \theta Q)^\alpha L_Y^{1-\alpha} \Rightarrow$$

$$\Rightarrow \underline{Y = A^{\frac{1}{1-\alpha}} (\alpha \theta Q)^{\frac{\alpha}{1-\alpha}} L_Y}$$

6°/ Innovation

- Recall that $V(\kappa_i)$ is a random variable because the timespan of rung κ_i is uncertain

- $$E\left[V(\kappa_i)\right] = \frac{\pi(\kappa_i)}{r + p(\kappa_i)},$$ where $p(\kappa_i)$ – probability density per unit of time,

or $$r = \frac{\pi(\kappa_i) - p(\kappa_i)\, EV(\kappa_i)}{EV(\kappa_i)}.$$

- Assume R&D technology:

  – $p(\kappa_i)$ depends only on total R&D expenditure $Z(\kappa_i)$:

$$p(\kappa_i) = \underbrace{Z(\kappa_i)}_{\text{LINEAR}} \cdot \underbrace{\phi(\kappa_i)}_{\text{EFFECT OF THE CURRENT TECH POSITION}}.$$

- Free entry into R&D implies

$$\underbrace{p(\kappa_i)\, EV(\kappa_i+1)}_{\text{NET EXPECTED RETURN PER UNIT OF TIME}} = \underbrace{Z(\kappa_i)}_{\text{COST}}$$

$$Z(\kappa_i)\,\phi(\kappa_i)\, E\big(V(\kappa_i+1)\big) = Z(\kappa_i) \qquad \big(Z(\kappa_i) > 0\big)$$

$$\phi(\kappa_i)\, EV(\kappa_i+1) = 1$$

$$r + p(\kappa_i+1) = \phi(\kappa_i)\, \underbrace{\bar{\pi}\, q^{(\kappa_i+1)\left(\frac{\theta}{1-\theta}\right)} \left(\frac{1-\theta}{\theta}\right)}_{\pi(\kappa_i+1)}.$$

Let us now make a simplifying assumption that

$$\phi(k_i) \equiv \frac{1}{\zeta}\, q^{-(k_i+1)\left(\frac{\theta}{1-\theta}\right)} \cdot \left(\frac{\theta}{1-\theta}\right), \quad \zeta > 0 \text{ — "COST OF DOING RESEARCH"}$$

In such case,

$$\underbrace{r + p(k_i+1) = \frac{\pi}{\zeta}}, \quad \text{the same for all } k_i.$$

And so $\underbrace{p = \frac{\pi}{\zeta} - r}$, implying

- $Z(k_i) = q^{(k_i+1)\left(\frac{\theta}{1-\theta}\right)}\left(\frac{1-\theta}{\theta}\right) \cdot \left(\pi - r\zeta\right)$ ← DISTRIBUTION OF R&D EXPENDITURES ACROSS SECTORS

- Aggregate R&D spending is

$$Z = \int_0^1 Z(k_i)\, di = \int_0^1 \underbrace{q^{\frac{\theta}{1-\theta}}\left(\frac{1-\theta}{\theta}\right)\left(\pi - r\zeta\right)} \cdot q^{k_i\left(\frac{\theta}{1-\theta}\right)} di =$$

$$= q^{\frac{\theta}{1-\theta}}\left(\frac{1-\theta}{\theta}\right)\left(\pi - r\zeta\right) Q^{\frac{\theta}{1-\theta}}.$$

7°/ Households

- Usual dynamic optimization problem implies

$$\underbrace{\frac{\dot{c}}{c} = \frac{r-\rho}{\gamma}}_{\text{EULER EQ.}} \qquad \left(\text{note that } \hat{C} = \hat{c} \text{ because } L \equiv \text{const}\right)$$

8°/ Dynamics

(for a very specific parametrization — KNIFE-EDGE ONE — discussed by Barro & Sala-i-Martin 2003)

→ NOTE AGGREGATE IDENTITIES:

- $Y = A^{\frac{1}{1-\alpha}} (\alpha\theta)^{\frac{\alpha}{1-\alpha}} L_Y \, Q^{\frac{\alpha}{1-\alpha}}$
- $X = \alpha\theta Y Q = (\alpha\theta)^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} L_Y \, Q^{\frac{1}{1-\alpha}}$
- $Z = q^{\frac{\theta}{1-\theta}} \left(\frac{1-\theta}{\theta}\right) (\bar{\pi} - r\zeta) \, Q^{\frac{\theta}{1-\theta}}$
- $\bar{\pi} = \left(\frac{\alpha\theta Y}{X^{\theta}}\right)^{\frac{1}{1-\theta}} = (\alpha\theta)^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} L_Y \cdot Q^{\frac{\alpha-\theta}{(1-\alpha)(1-\theta)}}$.

→ NOW ASSUME $\boxed{\alpha = \theta}$ $\boxed{!}$

THIS IMPLIES

$$Y \sim X^{\alpha} \sim Z \sim Q^{\frac{\alpha}{1-\alpha}}, \quad \text{and} \quad \bar{\pi} \equiv \text{const}.$$

→ IF $L_Y \equiv$ Const (fixed share of R&D employment), then:

$$\hat{Y} = \alpha\hat{X} = \hat{Z} = \frac{\alpha}{1-\alpha}\hat{Q} := g$$

(also, $\hat{C} = g$)

GROWTH RATE OF THE ECONOMY DRIVEN BY QUALITY INNOVATIONS

9°/ Dynamics of Q

$E(\Delta Q) = ?$

- Let's work with $Q^{\frac{\theta}{1-\theta}} = \int_0^1 q^{k_i\left(\frac{\theta}{1-\theta}\right)} di := \bar{Q}$

$$\rightarrow E(\Delta \bar{Q}) = \int_0^1 p(k_i) \cdot \left(q^{(k_i+1)\left(\frac{\theta}{1-\theta}\right)} - q^{k_i\left(\frac{\theta}{1-\theta}\right)} \right) di =$$

$$= \int_0^1 p \cdot \underbrace{\left(q^{\frac{\theta}{1-\theta}} - 1\right)}_{>0} q^{k_i\left(\frac{\theta}{1-\theta}\right)} di = p\left(q^{\frac{\theta}{1-\theta}} - 1\right)\bar{Q},$$

and therefore

$$E\left(\frac{\Delta \bar{Q}}{\bar{Q}}\right) = p\left(q^{\frac{\theta}{1-\theta}} - 1\right).$$

- Law of large numbers allows us to treat $\Delta \bar{Q}$ as deterministic:

$$\frac{\theta}{1-\theta}\hat{Q} = \hat{\bar{Q}} = p\left(q^{\frac{\theta}{1-\theta}} - 1\right) = \left(\frac{\pi}{\xi} - r\right)\left(q^{\frac{\theta}{1-\theta}} - 1\right).$$

10°/ Equilibrium rate of return r and growth rate g

$$\begin{cases} \hat{C} = g = \frac{r-\rho}{\gamma} \\ g = \left(\frac{\pi}{\xi} - r\right)\left(q^{\frac{\theta}{1-\theta}} - 1\right), \end{cases}$$

and assume $\alpha = \theta$

Note that r - fixed $\Rightarrow$ g - fixed $\Rightarrow$ NO TRANSITIONAL DYNAMICS, JUST BALANCED GROWTH!

Solving the system implies:

- $$g = \frac{\left(\frac{\bar{\pi}}{\zeta} - \rho\right)\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)}{1 + \gamma\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)}$$ ← GROWTH RATE OF THE ECONOMY

- $$r = \frac{\rho + \gamma \frac{\bar{\pi}}{\zeta}\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)}{1 + \gamma\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)},$$

where $\bar{\pi} = \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L_Y$.

This implies also a constant probability of innovation,

$$p = \frac{\frac{\bar{\pi}}{\zeta} - \rho}{1 + \gamma\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)}$$

Comments:

- determinants of long-run growth:
  - model parameters α, γ, ρ
  - technology level A $(A\uparrow \Rightarrow g\uparrow)$
  - population size L_Y $(L_Y\uparrow \Rightarrow g\uparrow)$ — SCALE EFFECT
  - size of the quality innovation rung, q $(q\uparrow \Rightarrow g\uparrow)$
- "Schumpeterian" flavor — creative destruction
- relies on a very specific parametrization $\alpha = \theta$, and of the function $\phi(k_i)$
- playing with $\phi(k_i)$ may destroy the asymptotically balanced growth property, but may also alleviate the scale effect.

Alternative definition of $\phi(\kappa_i)$:

$$\phi(\kappa_i) = \frac{1}{\zeta} \cdot \frac{1}{Y(\kappa_i+1)} = \frac{1}{\zeta \underbrace{A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}}}_{\text{CONST}} L_Y \, q^{(\kappa_i+1)\left(\frac{\alpha}{1-\alpha}\right)}}$$

!!

- Following analogous steps as before, we arrive at

$$g = \frac{\left(\frac{\alpha(1-\alpha)}{\zeta} - \rho\right)\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)}{1 + \gamma\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)}.$$

- This looks very similar to the previous version, but now there is no scale effect! A scale-free model.

Final notes:

— we have assumed throughout that $L_Y \equiv L$, and thus there was no competition for labor between the production and the R&D sector.

— we have skipped physical capital accumulation — the only asset available for households' savings are the shares of firms producing intermediate inputs $\{x_i\}_{i\in[0,1]}$.

— adding either of these two possibilities could be a source of transitional dynamics.